Week 4:

Electric Potential

Potential and Potential Energy: analogy to gravitation

RECAP

In the introductory mechanics course, we have seen that force on a particle of mass m located at a distance r from Earth's center due to the gravitational interaction between the particle and the Earth obeys an inverse-square law

$$\vec{\mathbf{F}}_g = -G\frac{Mm}{r^2}\hat{\mathbf{r}},$$

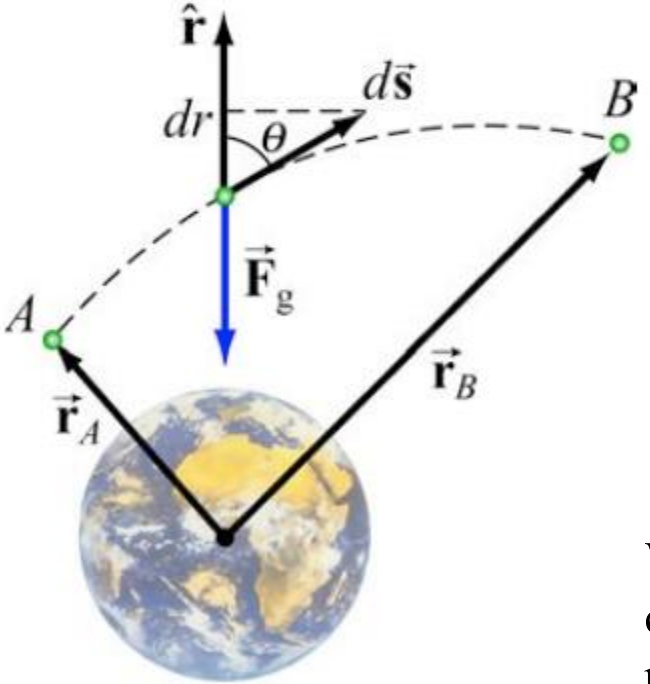
$\hat{\mathbf{r}}$ is a unit vector pointing radially outward from the Earth

$$G = 6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2}$$

The corresponding gravitation field

$$\vec{\mathbf{g}} = \frac{\vec{\mathbf{F}}_g}{m} = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

Consider moving a particle of mass *m* under the influence of gravity. The work done by gravity in moving m from A to B is



$$W_G = \int \vec{\mathbf{F}}_g\cdot d\vec{\mathbf{s}} = \int_{r_A}^{r_B}\left(-\frac{GMm}{r^2}\right)dr =$$

$$= \left[\frac{GMm}{r}\right]_{r_A}^{r_B} = GMm\left(\frac{1}{r_B}-\frac{1}{r_A}\right).$$

The result shows that W_G is **independent of the path** taken; it depends only on the endpoints A and B.

If the object returns to its starting point, then the work done by the gravitation force on the object is zero along this closed path.
Any force that satisfies this property for all closed paths is called a **conservative force**:

$$\oint_{\text{all closed paths}} \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} = 0 \quad \text{(conservative force).}$$

When dealing with a conservative force, it is often convenient to introduce the concept of change in potential energy function, $\Delta U = U_B - U_A$ between any two points in space, A and B

$$\Delta U = U_B - U_A = -\int_A^B \vec{\mathbf{F}}\cdot d\vec{\mathbf{s}} = -W$$

where W is the work done by the force on the object.

Electric Potential and Potential Energy

Our treatment of electrostatics will be similar to gravitation because the electrostatic force $\vec{F}_{el}$ also obeys an **inverse-square law**. In addition, it is also **conservative**. In the presence of an electric field $\vec{E}$, in analogy to the gravitational field $\vec{g}$, we define **the electric potential difference *ΔV* between two points A and B as**

*d***s** is the infinitesimal displacement vector, oriented tanged to a path through space. This path may be straight or curved, and an integral performed along this path is called either a *path or line integral*.

where q is a test charge.

$$\boxed{\begin{gathered}\Delta V = -\int_A^B (\vec{\mathbf{F}}_e / q)\cdot d\vec{\mathbf{s}} = -\int_A^B \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}\,, \\ \Delta V = V_{BA} = V_B - V_A\end{gathered}}$$

The potential difference *ΔV*, represents the negative of the work done per unit charge by the electrostatic force when a test charge q moves from points A to B.

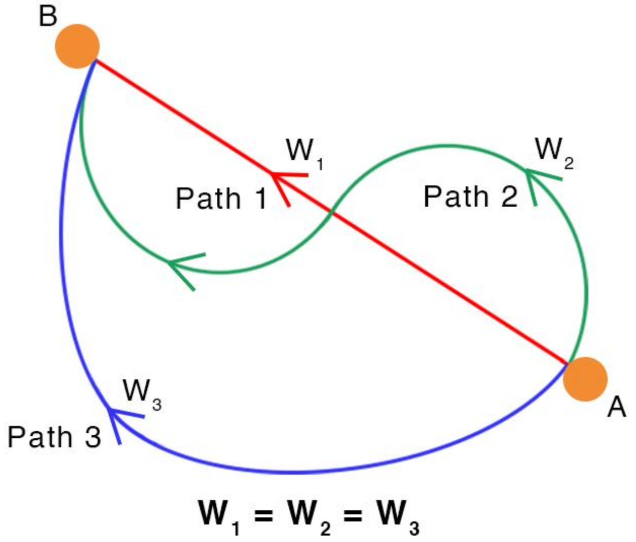
Again, electric potential difference *ΔV* should not be confused with electric potential energy difference *ΔU*.

Suppose an object with charge q is moved across a potential difference *ΔV*, the change in its potential energy is:

$$\boxed{\Delta U = U_B - U_A = -W = q\Delta V}$$

The electric force is a CONSERVATIVE force.

W does not depend on the path taken by the charge but only depends on the initial (A) and final (B) positions.

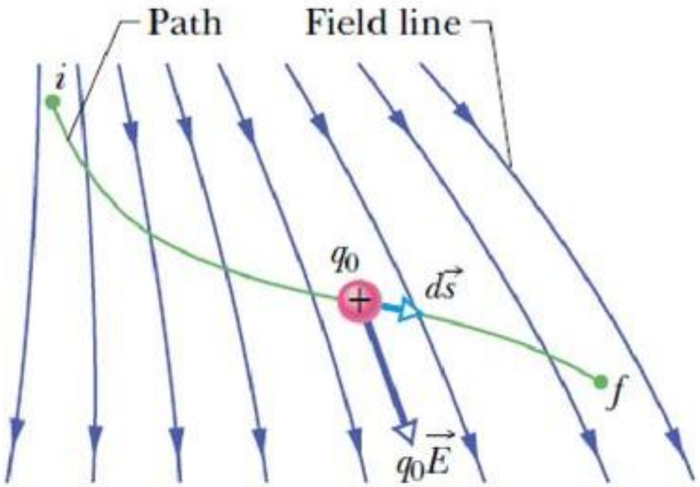


When a test charge q is moved (without changing the kinetic energy) in the electric field, electric force does work *W; or an external force should do work (–W) against the electric force.*

The change in potential energy between two points equals the negative of the work done by the conservative force as an object moves between the two points.

Calculating the Potential from the Field

To move a cherge in presence of an electric field work must be done against the electric force



A test charge q_0 moves from point i to point f along the path shown in a nonuniform electric field. During a displacement $d\vec{s}$, an electrostatic force $q_0\vec{E}$ acts on the test charge. This force points in the direction of the field line at the location of the test charge.

Infinitesimal work done by the field $dW = \vec{F}\cdot d\vec{s}.$

For the situation of Fig. $dW = q_0\vec{E}\cdot d\vec{s}.$

As this amount of work is done by the field, the potential energy of the charge–field system is changed by an amount

$$dU = -dW$$

Total work: $W = q_0 \int_i^f \vec{E}\cdot d\vec{s}. \;= \Delta U$

$$V_f - V_i = -\int_i^f \vec{E}\cdot d\vec{s}.$$

The potential difference $V_f - V_i$ between any two points ***i*** and ***f*** in an electric field is equal to the negative of the line integral from ***i*** to ***f***. Since the electrostatic force is conservative, all paths yield the same result. Just as with potential energy, **only differences in electric potential are meaningful**.

We often take the value of the electric potential to be zero at some convenient point in an electric field.

$$V = -\int_i^f \vec{E}\cdot d\vec{s},$$

In this way, we can define the potential V at any point in space ***f***, relative to the zero potential at point ***i***. If point ***i*** is at infinity, then this is the potential V at any point ***f*** relative to the zero potential at infinity.

Electric Potential: units

Definition:

Potential energy **U** of a unit positive charge is called

Electric Potential (Potential) $V = U/q$

- Potential is a scalar characteristic of an electric field
- SI unit of Potential is Volt: 1 ***V*** = 1 Joule/1 Coulomb= 1 J/C

Potential difference: $\Delta V = V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l}$ $\Delta V = W/q$

- Units of Electric field is N/C, but also V/m =>

  Electric field is a measure of gradient (rate of changing) of Electric Potential.

- When dealing with systems at the atomic or molecular scale, a joule [J] often turns out to be too large as an energy unit.
- A more useful scale is electron volt [eV], Electron volt, **eV**, is a unit of energy:
- energy to move an electron through ΔV=1 V:

$$1\ \mathrm{eV} = 1.60 \times 10^{-19}\ \mathrm{C \cdot V} = 1.60 \times 10^{-19}\ \mathrm{J}$$

Electrostatic potential: relations with the Electric field

The static electric field "**E**" is conservative ⇒

there exists a scalar field $'V'$ such that: $\boxed{\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})} = \left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) V(\mathbf{r}) \quad \forall \mathbf{r}$

$$\Rightarrow \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = \int_{\mathbf{r}_0}^{\mathbf{r}} -\nabla V \cdot d\mathbf{l} \; = \; V(\mathbf{r}_0) - V(\mathbf{r})$$

$$\Rightarrow \boxed{V(\mathbf{r}) - V(\mathbf{r}_0) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}} \quad \Rightarrow \quad \boxed{V(\mathbf{r}) = V(\mathbf{r}_0) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}}$$

Notes:

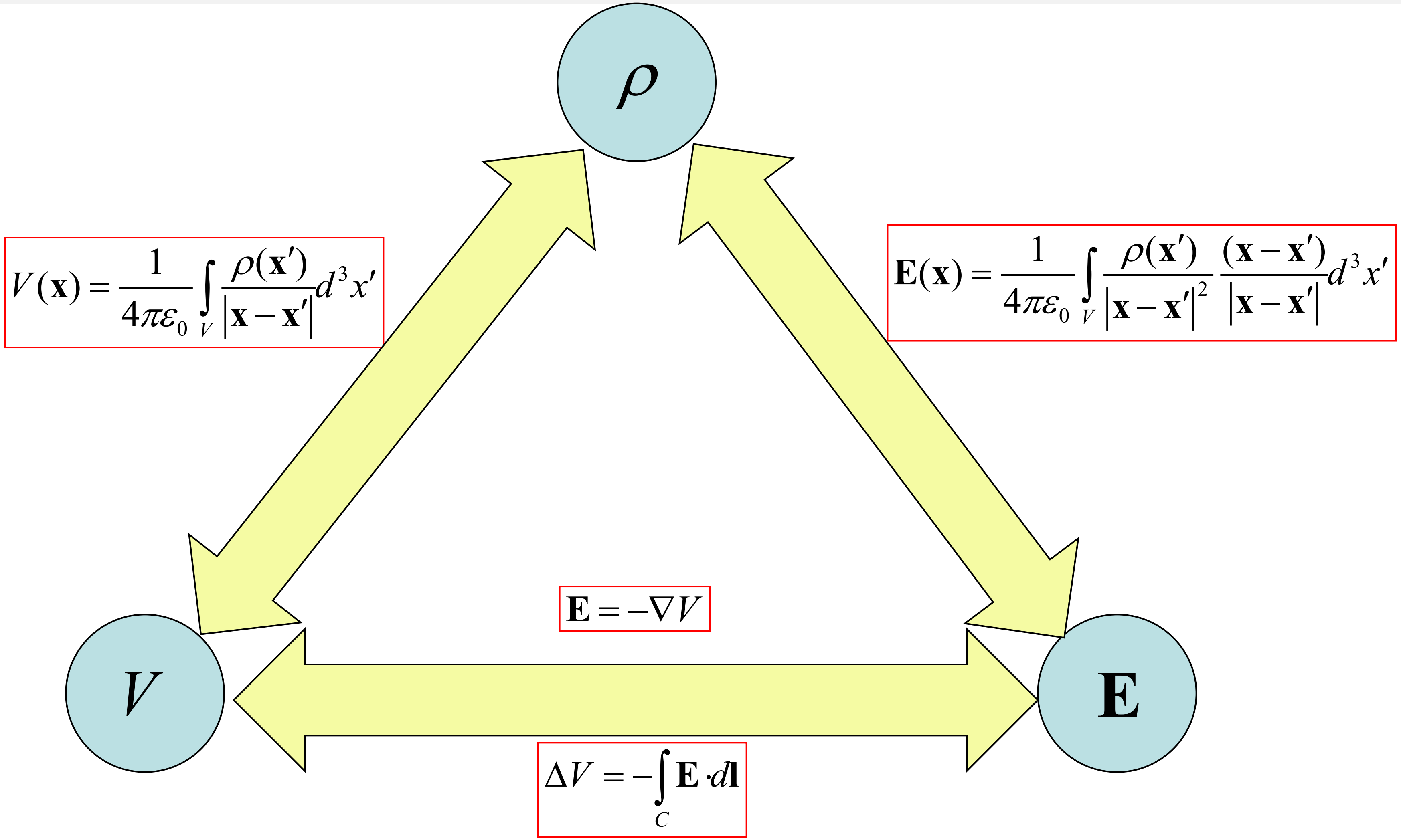
1) The electrical potential $V(\mathbf{r})$ is also often indicated as $\Phi(\mathbf{r})$ or $\varphi(\mathbf{r})$.

2) The electrical potential $V(\mathbf{r})$is a continuous scalar function defined up to a constant $V(\mathbf{r}_0)$. Supposing $V(\mathbf{r}_0) = 0 \Rightarrow V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$

Attention: the chioce $of \; V(\mathbf{r} = \infty)$=0 is not possible if the charge density at infinity is non−zero
(e.g..: charged infinite wire; charged infinite plane,...)

Thus, if our real charge system can be approximated by a **charge system located in specific area of space, then we can consider the potential at infinity to be zero**. However, there are situations where the charge system can be approximated (to simplify the calculations) by a charge distribution extending to infinity. In such a case, we cannot consider that the potential at infinity is zero.

Summary of electrostatics



$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

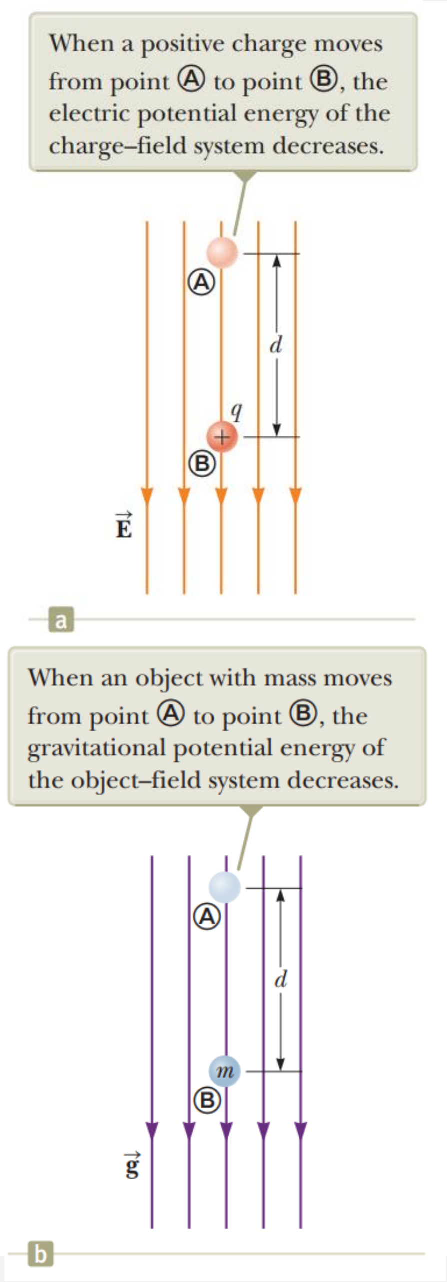
$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$\mathbf{E} = -\nabla V$$

$$\Delta V = -\int_C \mathbf{E}\cdot d\mathbf{l}$$

Note: Electrostatics can be divided into:
Summation problems: the charge density is specified at each point in space and the problem of finding the electric field is reduced to performing an integral.
Problem of limit value: the charge density cannot be specified once and for all at every point in space. This happens when matter is present because the Lorentz force induces the redistribution of charge density inside matter until equilibrium is reached. The electric field can be uniquely determined everywhere, provided you specify a model of matter (behavior of the electric field inside matter) and boundary conditions for Maxwell's differential equations.

Potential Difference in a Uniform Electric Field



Let us calculate the potential difference between two points A and B separated by a distance d

$$V_{Ⓑ} - V_{Ⓐ} = \Delta V = -\int_{Ⓐ}^{Ⓑ} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\int_{Ⓐ}^{Ⓑ} E\,ds\,(\cos 0°) = -\int_{Ⓐ}^{Ⓑ} E\,ds$$

Because ***E*** is constant, we can remove it from the integral sign; this gives:

$$\Delta V = -E\int_{Ⓐ}^{Ⓑ} ds$$

$$\Delta V = -Ed$$

The negative sign indicates that the electric potential at point B is lower than at point A; that is, $V_B < V_A$. **Electric field lines always point in the direction of decreasing electric potential.**

Now suppose that a test charge q_0 moves from A to B. We can calculate the change in the potential energy of the charge–field system

$$\Delta U = q_0\,\Delta V = -q_0 E d$$

- We see that if q_0 is positive, then ΔU is negative. We conclude that **a system consisting of a positive charge and an electric field loses electric potential energy when the charge moves in the direction of the field.** This means that an electric field does work on a positive charge when the charge moves in the direction of the electric field.
- If a positive test charge is released from rest in this electric field, it experiences an electric force $q_0\boldsymbol{E}$ in the direction of ***E*** (downward in Fig.). Therefore, it accelerates downward, gaining kinetic energy. **As the charged particle gains kinetic energy, the charge–field system loses an equal amount of potential energy.**
- If q_0 is negative, then ΔU is positive: **A system consisting of a negative charge and an electric field gains electric potential energy when the charge moves in the direction of the field.**

Electric Potential Energy Difference

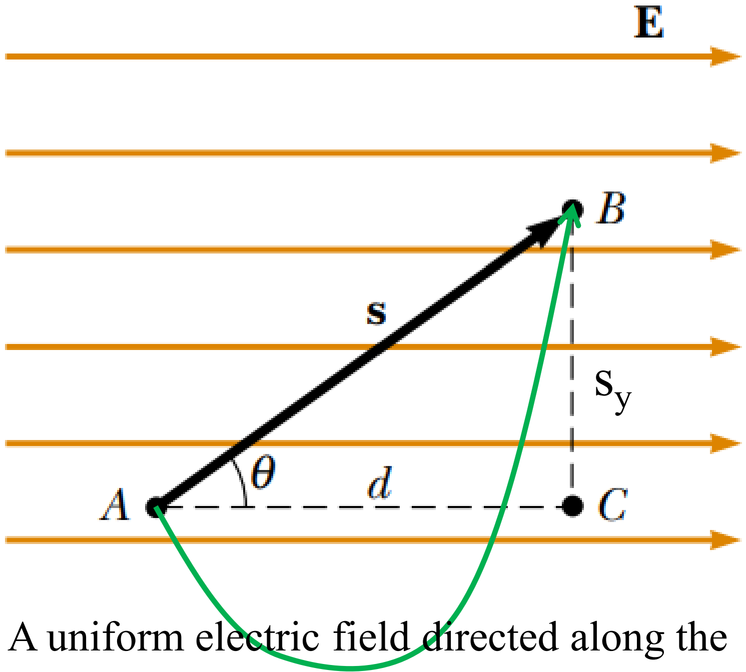
Now consider the more general case of a charged particle that moves between A and B in a uniform electric field such that the vector **s** is not parallel to the field lines

The external work (per unit charge) for moving a charge from A to B is equal to the difference of electric potential between points B and A in space

$$\Delta V = -\int_A^B \mathbf{E}\cdot d\mathbf{s} = -\mathbf{E}\cdot\int_A^B d\mathbf{s} = -\mathbf{E}\cdot\mathbf{s}$$

The change in potential energy of the charge–field system is

$$U_B - U_A = \Delta U = q_0\,\Delta V = -q_0\,\mathbf{E}\cdot\mathbf{s}$$



A uniform electric field directed along the positive x axis. Point B is at a lower electric potential than point A. **Points B and C are at the same electric potential.**

When a charge in electric field is diplaced against the Coulomb force, the Potential Energy increases

When a charge in electric field is diplaced by the Coulomb force, the Potential Energy decreases

$$\Delta V_{BA} = \Delta V_{CA} + \Delta V_{BC}$$
$$= -q_0\cdot E\cdot d + 0$$

Any system spontaneously moves into a new configuration to minimize the potential energy

Summary: Definition of Electric Potential and Potential Energy

Electric Potential Energy difference:

$$\Delta U_{BA} \equiv U_B - U_A = W^{ext}_{A\to B} = -q\int_A^B \vec{E}\cdot d\vec{l}$$

Electric Potential Energy:

$$U_B \equiv U_A + \Delta U_{BA};$$

$U_A \equiv 0$ (e.g., $A \to \infty$ for not infinite charge distributions)

$$U_B = \Delta U_{B\infty} = W^{ext}_{\infty B} = -q\int_\infty^B \vec{E}\cdot d\vec{l} = q\int_B^\infty \vec{E}\cdot d\vec{l}$$

Electric Potential:

$$V_B = \frac{U_B}{q} = \int_B^\infty \vec{E}\cdot d\vec{l};$$

Electric Potential difference:

$$\Delta V_{BA} \equiv V_B - V_A = \frac{\Delta U_{BA}}{q};$$

The Electric Field Between Two Parallel Plates of Opposite Charge

A battery produces a specified potential difference ΔV between conductors attached to the battery terminals. A 12-V battery is connected between two parallel plates, as shown in Figure The separation between the plates is $d = 0.30$ cm, and we assume the electric field between the plates to be uniform. (This assumption is reasonable if the plate separation is small relative to the plate dimensions and if we do not consider locations near the plate edges.) Find the magnitude of the electric field between the plates.

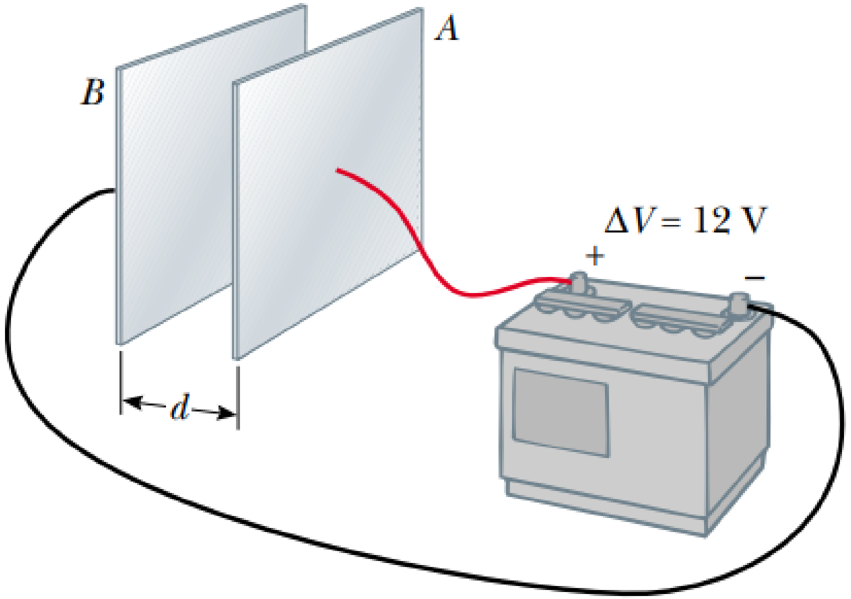


Figure A 12-V battery connected to two parallel plates. The electric field between the plates has a magnitude given by the potential difference ΔV divided by the plate separation d.

Solution The electric field is directed from the positive plate (A) to the negative one (B), and the positive plate is at a higher electric potential than the negative plate is. The potential difference between the plates must equal the potential difference between the battery terminals. We can understand this by noting that all points on a conductor in equilibrium are at the same electric potential[1]; no potential difference exists between a terminal and any portion of the plate to which it is connected. Therefore, the magnitude of the electric field between the plates is

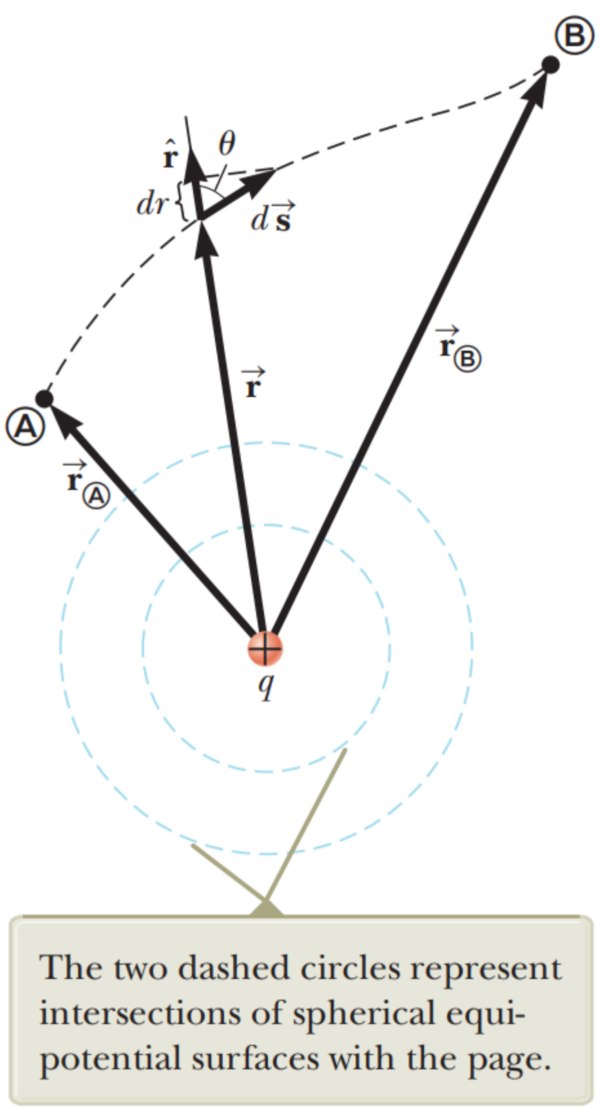
$$E = \frac{|V_B - V_A|}{d} = \frac{12\ \text{V}}{0.30 \times 10^{-2}\ \text{m}} = 4.0 \times 10^{3}\ \text{V/m}$$

The configuration of plates in Figure 25.5 is called a *parallel-plate capacitor*,

---

1) The electric field vanishes within a conductor in electrostatic equilibrium; thus, the path integral between any two points in the conductor must be zero

Electric Potential and Potential Energy Due to Point Charges



The two dashed circles represent intersections of spherical equipotential surfaces with the page.

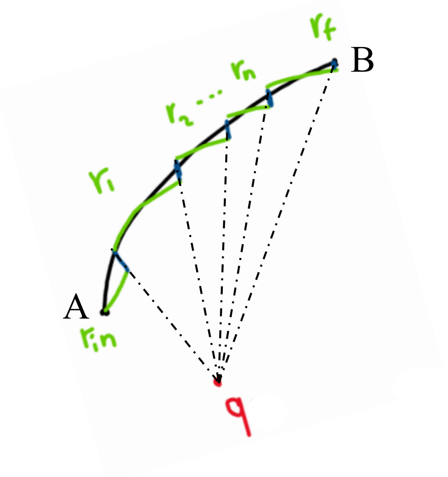
$$V_{\text{Ⓑ}} - V_{\text{Ⓐ}} = -\int_{\text{Ⓐ}}^{\text{Ⓑ}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

$$\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = k_e \frac{q}{r^2}\,\hat{\mathbf{r}} \cdot d\vec{\mathbf{s}}$$

The potential of the field is the work to bring a unit positive charge from A to B.

Because the magnitude of $\hat{\mathbf{r}}$ is 1, the dot product $\hat{\mathbf{r}} \cdot d\vec{\mathbf{s}} = ds\cos\theta$, where θ is the angle between $\hat{\mathbf{r}}$ and $d\vec{\mathbf{s}}$. Furthermore, $ds\cos\theta$ is the projection of $d\vec{\mathbf{s}}$ onto $\hat{\mathbf{r}}$; therefore, $ds\cos\theta = dr$. That is, any displacement $d\vec{\mathbf{s}}$ along the path from point Ⓐ to point Ⓑ produces a change dr in the magnitude of $\vec{\mathbf{r}}$, the position vector of the point relative to the charge creating the field. Making these substitutions, we find that $\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = (k_e q/r^2)\,dr$; hence, the expression for the potential difference becomes

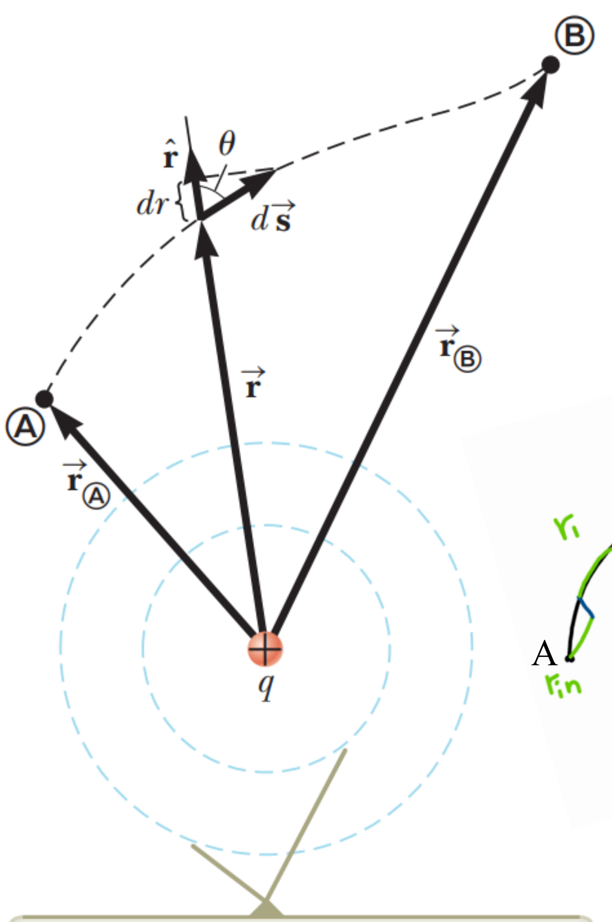
Figure 25.7 The potential difference between points Ⓐ and Ⓑ due to a point charge q depends *only* on the initial and final radial coordinates $r_{\text{Ⓐ}}$ and $r_{\text{Ⓑ}}$.



Let's approximate this path from A to B in smaller pieces:

1) Some along radial directions;
2) Some along the segment of arc with radius r_i;

Electric Potential and Potential Energy Due to Point Charges



The two dashed circles represent intersections of spherical equipotential surfaces with the page.

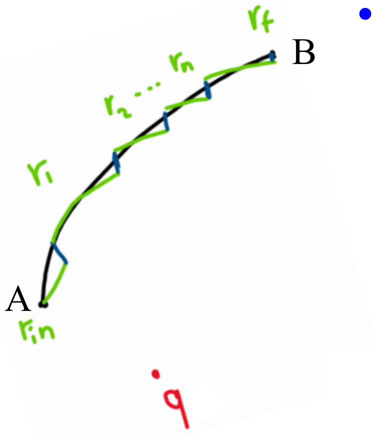
Figure 25.7 The potential difference between points Ⓐ and Ⓑ due to a point charge q depends *only* on the initial and final radial coordinates $r_Ⓐ$ and $r_Ⓑ$.

$$V_Ⓑ - V_Ⓐ = -\int_Ⓐ^Ⓑ \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

$$\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = k_e \frac{q}{r^2} \hat{\mathbf{r}} \cdot d\vec{\mathbf{s}}$$

The potential of the field is the work to bring a unit positive charge from A to B.

- Electric field is radial
- Potential does NOT depend on the path A->B
- Only radial displacement contributes to ΔV



$$V_Ⓑ - V_Ⓐ = -k_e q \int_{r_Ⓐ}^{r_Ⓑ} \frac{dr}{r^2} = k_e \frac{q}{r}\bigg|_{r_Ⓐ}^{r_Ⓑ}$$

$$V_Ⓑ - V_Ⓐ = k_e q \left[\frac{1}{r_Ⓑ} - \frac{1}{r_Ⓐ}\right]$$

- Electric potential is a <u>relative</u> quantity; only ΔV is meaningful

Conventionally, for a finite charge distribution the potential at infinity $r_A \rightarrow \infty$ $V_A = 0$.

$$V = 0 \text{ at } r_Ⓐ = \infty \qquad \boxed{V = k_e \frac{q}{r}}$$

Comparison between Gravitational and Electric fields

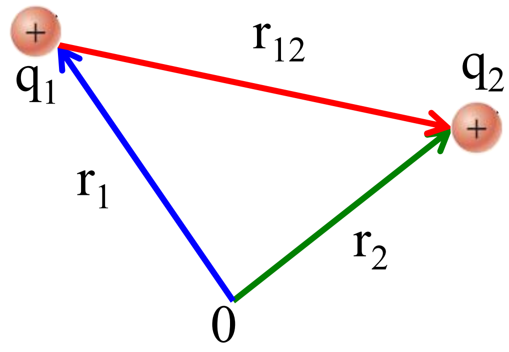
Gravity	Electrostatics
Mass m	Charge q
Gravitation force $\vec{\mathbf{F}}_G = -G\frac{Mm}{r^2}\hat{\mathbf{r}}$	Electric force $\vec{\mathbf{F}}_e = k_e\frac{Qq}{r^2}\hat{\mathbf{r}}$
Gravitation field $\vec{\mathbf{g}} = \vec{\mathbf{F}}_g / m$	Electric field $\vec{\mathbf{E}} = \vec{\mathbf{F}}_e / q$
Potential energy change $\Delta U = -\int_A^B \vec{\mathbf{F}}_G \cdot d\vec{\mathbf{s}}$	Potential energy change $\Delta U = -\int_A^B \vec{\mathbf{F}}_e \cdot d\vec{\mathbf{s}}$
Gravitational potential $\Delta V_G = -\int_A^B \vec{\mathbf{g}} \cdot d\vec{\mathbf{s}}$	Electric Potential $\Delta V = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$
Potential function, $V_G(\infty) = 0 : V_G = -\frac{GM}{r}$	Potential function, $V(\infty) = 0 : V = k_e\frac{Q}{r}$
$\lvert \Delta U_g \rvert = mgd$, (constant $\vec{\mathbf{g}}$)	$\lvert \Delta U \rvert = qEd$, (constant $\vec{\mathbf{E}}$)

Electric Potential Energy of two or more-point charges

Electric potential energy of a charge ***q_2***
in the field of ***q_1*** is:

$$U_2 = \Delta U_2(\infty \rightarrow r_2) = q_2 V(r_2, q_1) = k_e \frac{q_1 q_2}{r_{12}}$$

we put U=0 at infinity



$$V(r_2, q_1) = k_e \frac{q_1}{r_{12}}$$

This is how much work is to be done to move $\boldsymbol{q}_2$ from *infinite* to $\boldsymbol{r}_2$

Electric potential energy ***U*** is invariant relative to ***q_1*** and ***q_2*** :
one charge creates the field, the other one owns
the electric potential energy due to this field.

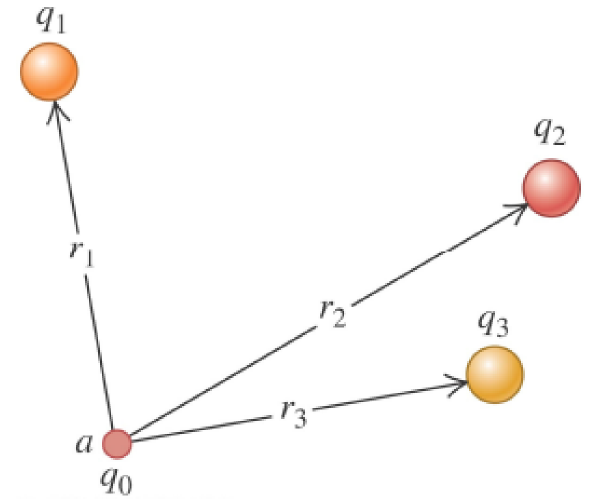
$$U_2 = q_2 V(r_2, q_1) = q_1 V(r_1, q_2) = U_1$$

Electric Potential Energy of two or more-point charges

Now let's consider the 3 charges (represented here on the right).
What is the change of potential energy (=work to be done)
to bring a new charge q_0 in the position *a* ?

The potential energy associated with q_0 at "a"
is the algebraic sum of U associated with each pair of charges.

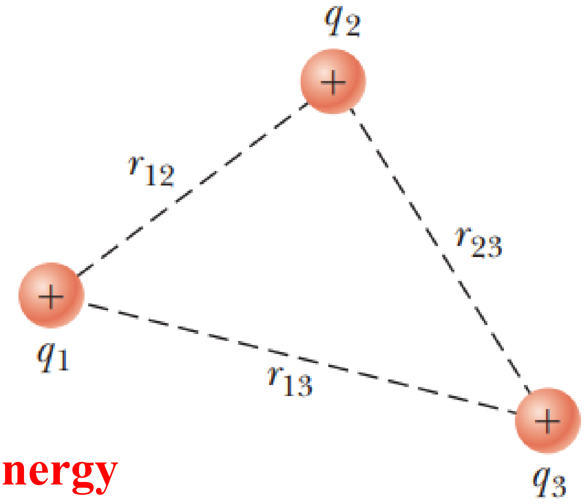
$$U = \frac{q_0}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \qquad\right) = \frac{q_0}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}$$



Let's now calculate the TOTAL potential energy
of a system of charges (i.e. this configuration on the right).

If the system consists, for example, of three charged particles, we can obtain the total potential energy of the system by calculating U for every pair of charges and summing the terms algebraically

$$U = k_e\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$



The total system potential energy
is the sum of all the contributions from distinct pairs.

Summary: Electric Potential of two or more-point charges

Electric Potential Energy of a System of Charges

Electric potential energy comes from the interaction between pairs of charged particles, so you have to add the potential energies of **each pair** of charged particles in the system.

(Could be a pain to calculate!)

Electric Potential of a System of Charges

The potential due to a particle depends only on the charge of that particle and where it is relative to some reference point.

The electric potential of a system of charges is simply the sum of the potential of each charge. (Much easier to calculate!)

$$V = \frac{U}{q_0} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i}$$

The Electric Potential Due to Two Point Charges

As shown in Figure 25.10a, a charge $q_1 = 2.00\ \mu\text{C}$ is located at the origin and a charge $q_2 = -6.00\ \mu\text{C}$ is located at (0, 3.00) m.

(A) Find the total electric potential due to these charges at the point *P*, whose coordinates are (4.00, 0) m.

$$V = k_e \sum_i \frac{q_i}{r_i}$$

$$V_P = k_e\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right)$$

$$V_P = (8.988 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)\left(\frac{2.00 \times 10^{-6}\ \text{C}}{4.00\ \text{m}} + \frac{-6.00 \times 10^{-6}\ \text{C}}{5.00\ \text{m}}\right)$$

$$= \boxed{-6.29 \times 10^3\ \text{V}}$$

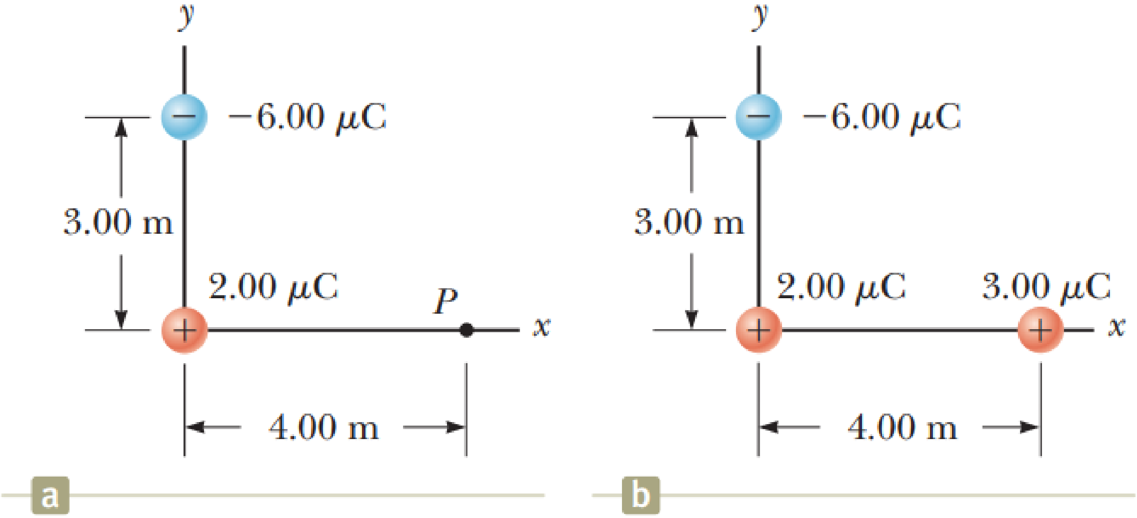


Figure 25.10 (Example 25.3) (a) The electric potential at *P* due to the two charges q_1 and q_2 is the algebraic sum of the potentials due to the individual charges. (b) A third charge $q_3 = 3.00\ \mu\text{C}$ is brought from infinity to point *P*.

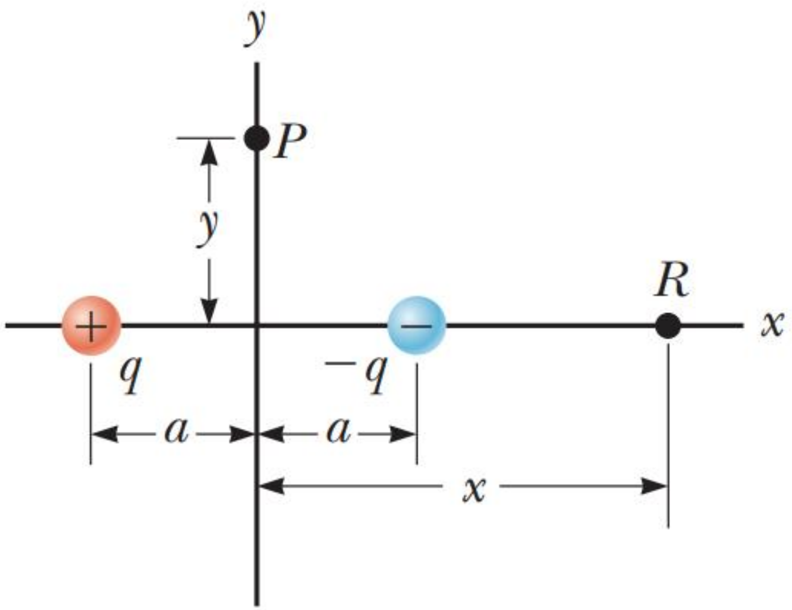
(B) Find the change in potential energy of the system of two charges plus a third charge $q_3 = 3.00\ \mu\text{C}$ as the latter charge moves from infinity to point *P* (Fig. 25.10b).

$$U_f = q_3 V_P$$

$$\Delta U = U_f - U_i = q_3 V_P - 0 = (3.00 \times 10^{-6}\ \text{C})(-6.29 \times 10^3\ \text{V})$$

$$= \boxed{-1.89 \times 10^{-2}\ \text{J}}$$

The Electric Potential Due to a Dipole



An electric dipole consists of two charges of equal magnitude and opposite sign separated by a distance 2a as shown in Figure. The dipole is along the x axis and is centered at the origin.

(A) Calculate the electric potential at point P on the y axis

$$V_P = k_e \sum_i \frac{q_i}{r_i} = k_e\left(\frac{q}{\sqrt{a^2+y^2}} + \frac{-q}{\sqrt{a^2+y^2}}\right) = \boxed{0}$$

(B) Calculate the electric potential at point R on the positive x axis.

$$V_R = k_e \sum_i \frac{q_i}{r_i} = k_e\left(\frac{-q}{x-a} + \frac{q}{x+a}\right) = \boxed{-\frac{2k_e qa}{x^2-a^2}}$$

(C) Calculate *V* and *Ex* at a point on the x axis far from the dipole

$$V_R = \lim_{x>>a}\left(-\frac{2k_e qa}{x^2-a^2}\right) \approx \boxed{-\frac{2k_e qa}{x^2} \quad (x>>a)}$$

$$E_x = -\frac{dV}{dx} = -\frac{d}{dx}\left(-\frac{2k_e qa}{x^2}\right)$$

$$= 2k_e qa\,\frac{d}{dx}\left(\frac{1}{x^2}\right) = \boxed{-\frac{4k_e qa}{x^3} \quad (x>>a)}$$

A dipole has a total charge of zero, so we expect that far from the dipole, the potential would decrease with increasing distance from the dipole more rapidly than it would for a configuration that has a nonzero net charge.

Obtaining the Value of the Electric Field from the Electric Potential

We now show how to calculate the value of the electric field if the electric potential is known in a certain region.

From $V_{Ⓑ} - V_{Ⓐ} = -\int_{Ⓐ}^{Ⓑ} \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ we can express the potential difference dV between two points a distance ds apart as

$$dV = -\mathbf{E} \cdot d\mathbf{s}$$

If the electric field has only one component E_x, then $\mathbf{E} \cdot d\mathbf{s} = E_x\, dx.$
Therefore, the above Equation becomes $dV = -E_x\, dx,$ $\xrightarrow{\quad}$ $E_x = -\dfrac{dV}{dx}$

The x component of the electric field is equal to the negative of the derivative of the electric potential with respect to x. Similar statements can be made about the y and z components.

In general, the electric potential is a function of all three spatial coordinates. If $V(r)$ is given in terms of the Cartesian coordinates, the electric field components E_x, E_y, and E_z can readily be found from $V(x, y, z)$ as the partial derivatives:

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z} \quad \xrightarrow{\quad} \quad \vec{E} = -\vec{\nabla} V$$

$$\vec{E} = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)V$$

Example : Calculating Electric Field from Electric Potential

Suppose the electric potential due to a certain charge distribution can be written in Cartesian Coordinates as

$$V(x,y,z) = Ax^2y^2 + Bxyz$$

where A, B and C are constants. What is the associated electric field?

Solution: The electric field can be found by using Eq. $E_x = -\dfrac{\partial V}{\partial x}$, $E_y = -\dfrac{\partial V}{\partial y}$, $E_z = -\dfrac{\partial V}{\partial z}$.

$$E_x = -\frac{\partial V}{\partial x} = -2Axy^2 - Byz$$

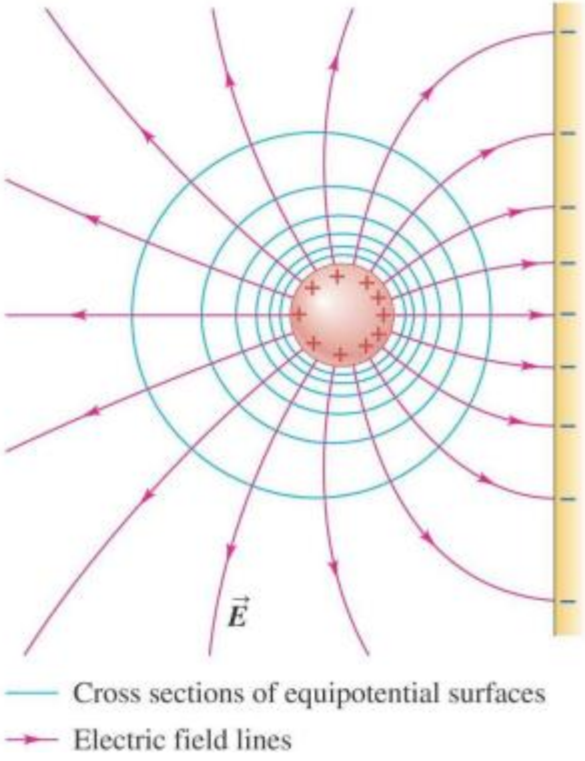
$$E_y = -\frac{\partial V}{\partial y} = -2Ax^2y - Bxz$$

$$E_z = -\frac{\partial V}{\partial z} = -Bxy$$

Therefore, the electric field is $\vec{\mathbf{E}} = (-2Axy^2 - Byz)\,\hat{\mathbf{i}} - (2Ax^2y + Bxz)\,\hat{\mathbf{j}} - Bxy\,\hat{\mathbf{k}}$.

Equipotential Surfaces

- **The points in space that are at the same electric potential V form an equipotantial surface**
- Moving a charge on an equipotential surface requires NO work; the electric potential energy ($U=qV$) is constant.
- At every point of the equipotential surface, the electric field is necessarily perpendicular to the equipotential surface (to the tangent of the surfece at that point).



- Field lines (curves) → E tangent

- Equipotential surfaces (curved surfaces) → E perpendicular

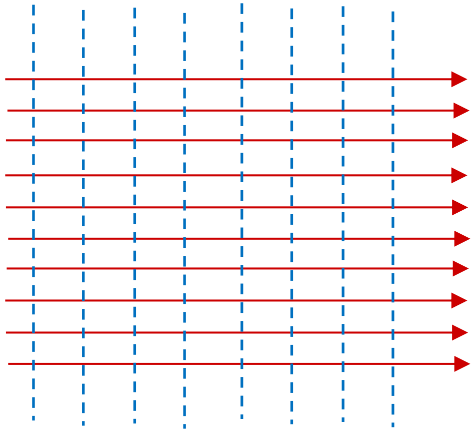
- **Field lines and equipotential surfaces are mutually perpendicular.**

At each crossing of an equipotential and field line, the two are perpendicular.

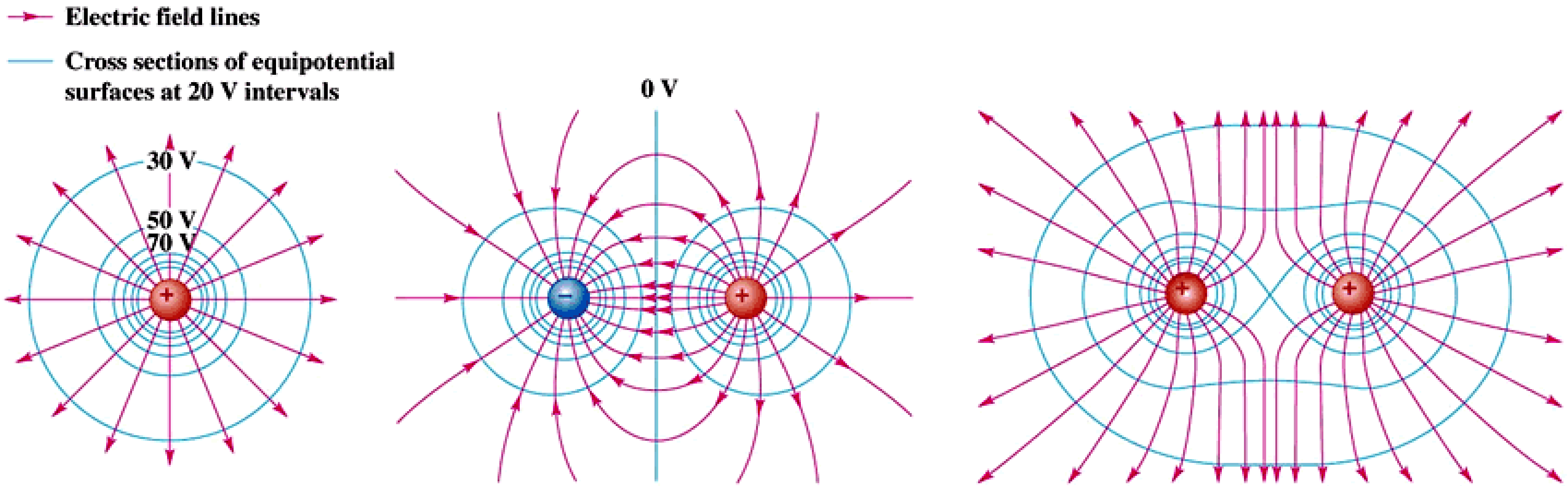
- If electric field uniform → field lines straight, parallel and equally spaced.
(see next slide) equipotentials → parallel planes perp. field lines.

Density of equipotential lines reflects grandient of Electric Potential and, therefore, strength of the Electric field

Equipotential Surfaces



Uniform electric field



(a) A single positive charge **(b) An electric dipole** **(c) Two equal positive charges**

The properties of equipotential surfaces can be summarized as follows:

(i) The electric field lines are perpendicular to the equipotentials and point from higher to lower potentials.

(ii) By symmetry, the equipotential surfaces produced by a point charge form a family of concentric spheres, and for constant electric field, a family of planes perpendicular to the field lines.

(iii) The tangential component of the electric field along the equipotential surface is zero, otherwise non-vanishing work would be done to move a charge from one point on the surface to the other.

(iv) No work is required to move a particle along an equipotential surface.

- **Important:** E does not need to be constant over an equipotential surface. Only V is constant.

Electric Potential Due to Continuous Charge Distributions

The electric potential *dV* at some point *P* in space due to the charge element *dq* is

(1)

$$dV = k_e \frac{dq}{r}$$

where *r* is the distance from the charge element to point *P*.

For the total potential at point *P*, we integrate Equation above for *dV* to include contributions from all elements of the charge distribution. Because each element is, in general, at a different distance from point *P* and k_e is constant, we can express the total electric potential *V* as

$$V = k_e \int \frac{dq}{r}$$

Note that this expression for *V* uses a particular reference: *the electric potential is taken to be zero when point P is infinitely far from the charge distribution.*

For many problems, it is possible in performing the integration to express *dq* and *r* in terms of a single variable. **To simplify the integration, consider the geometry involved in the problem.**

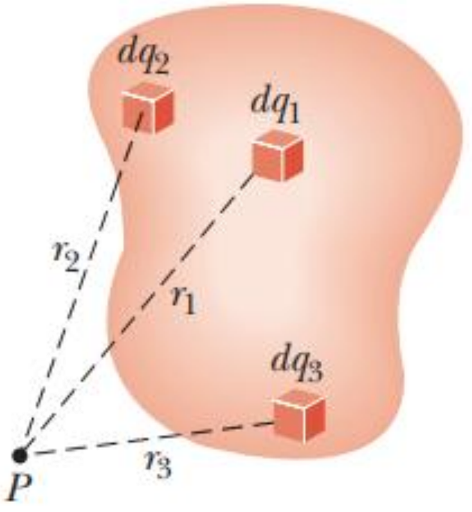


Figure 25.12 The electric potential at point *P* due to a continuous charge distribution can be calculated by dividing the charge distribution into elements of charge *dq* and summing the electric potential contributions over all elements. Three sample elements of charge are shown.

(2) Another method that you can use to obtain the electric potential due to a finite continuous charge distribution is to start with the definition of potential difference.

$$\Delta V \equiv \frac{\Delta U}{q_0} = -\int_A^B \mathbf{E} \cdot d\mathbf{s}$$

If the charge distribution has sufficient symmetry, we first evaluate **E** at any point using Gauss's law and then substitute the value obtained into the above Equation to determine the potential difference ΔV between any two points. **We then choose the electric potential *V* to be zero at some convenient point.**

Electric Potential Due to a Uniformly Charged Ring

(A) Find an expression for the electric potential at a point P located on the perpendicular central axis of a uniformly charged ring of radius a and total charge Q

$$V = k_e \int \frac{dq}{r} = k_e \int \frac{dq}{\sqrt{a^2 + x^2}}$$

$$V = \frac{k_e}{\sqrt{a^2 + x^2}} \int dq = \boxed{\frac{k_e Q}{\sqrt{a^2 + x^2}}}$$

(B) Find an expression for the magnitude of the electric field at point P

$$E_x = -\frac{dV}{dx} = -k_e Q \frac{d}{dx}(a^2 + x^2)^{-1/2}$$

$$= -k_e Q(-\tfrac{1}{2})(a^2 + x^2)^{-3/2}(2x)$$

$$E_x = \boxed{\frac{k_e x}{(a^2 + x^2)^{3/2}} Q}$$

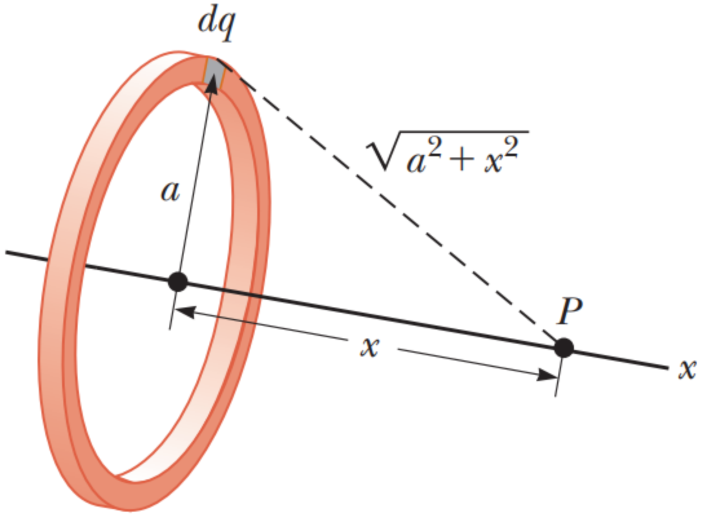


Figure 25.14 (Example 25.5) A uniformly charged ring of radius a lies in a plane perpendicular to the x axis. All elements dq of the ring are the same distance from a point P lying on the x axis.

Electric Potential Due to a Uniformly Charged Disk

A uniformly charged disk has radius R and surface charge density σ.

(A) Find the electric potential at a point P along the perpendicular central axis of the disk.

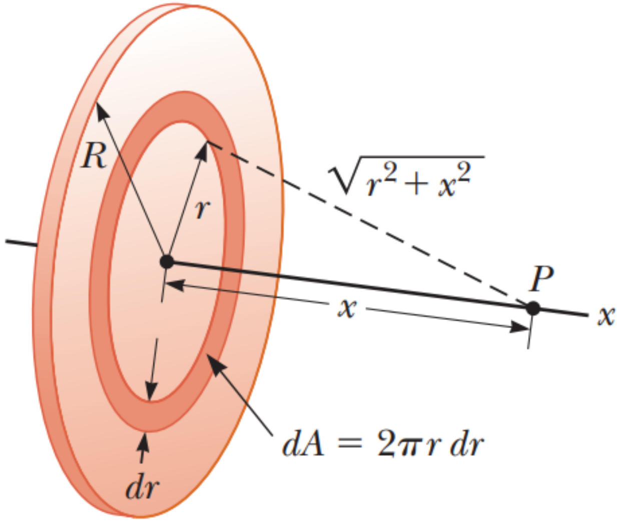
$$dq = \sigma\, dA = \sigma(2\pi r\, dr) = 2\pi\sigma r\, dr$$

$$dV = \frac{k_e\, dq}{\sqrt{r^2 + x^2}} = \frac{k_e 2\pi\sigma r\, dr}{\sqrt{r^2 + x^2}}$$

$$V = \pi k_e \sigma \int_0^R \frac{2r\, dr}{\sqrt{r^2 + x^2}} = \pi k_e \sigma \int_0^R (r$$

$$V = 2\pi k_e \sigma[(R^2 + x^2)^{1/2} - x]$$

Figure 25.15 (Example 25.6) A uniformly charged disk of radius R lies in a plane perpendicular to the x axis. The calculation of the electric potential at any point P on the x axis is simplified by dividing the disk into many rings of radius r and width dr, with area $2\pi r\, dr$.

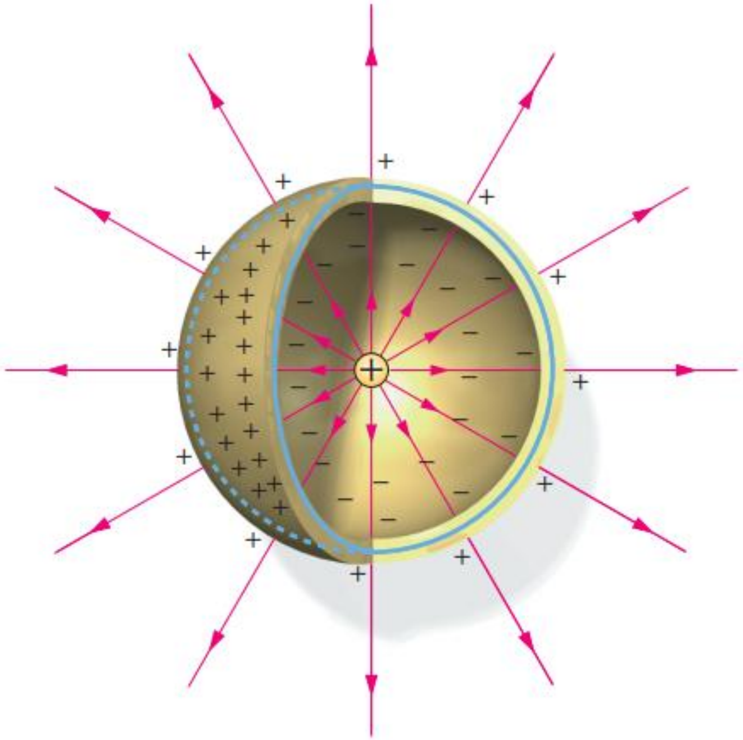


(B) Find the x component of the electric field at a point P along the perpendicular central axis of the disk.

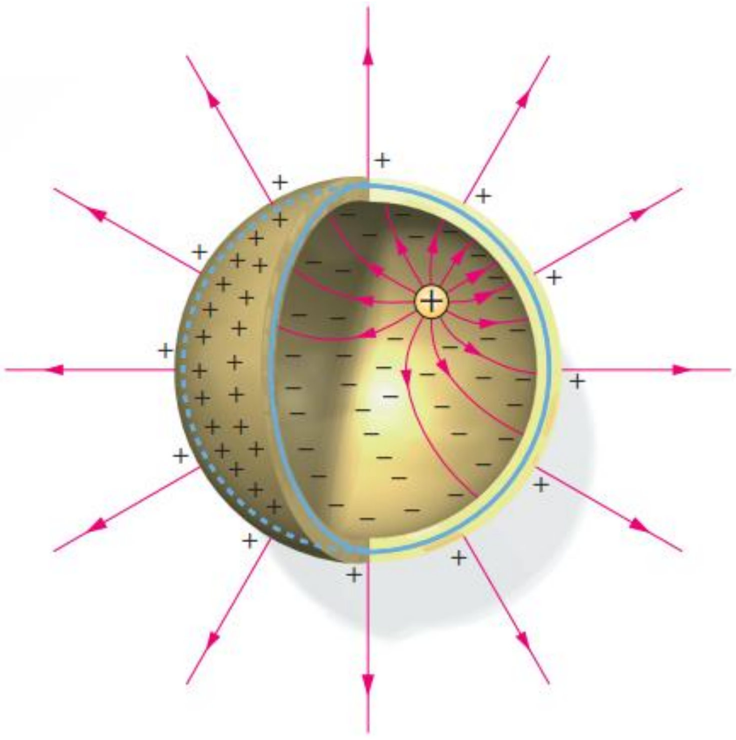
$$E_x = -\frac{dV}{dx} = 2\pi k_e \sigma \left[1 - \frac{x}{(R^2 + x^2)^{1/2}}\right]$$

Distribution of the free charges and electric field on a conductor

Electric field lines for an oppositely charged cylinder and plate, shown by bits of fine thread suspended in oil. Note that the field lines are normal to the surfaces of the conductors and that there are no lines inside the cylinder. The region inside the cylinder is electrically shielded from the region outside the cylinder



A point charge in the cavity at the center of a thick spherical conducting shell. Because the net charge within the Gaussian surface (indicated in blue) must be zero, we know a surface charge is induced on the inner surface of the shell, and because the conductor is neutral, an equal but opposite charge is induced on the outer surface. Electric field lines begin on the point charge and end on the inner surface. Field lines begin again on the outer surface.



The same conductor as in panel on the left with the point charge moved away from the center of the sphere. The charge on the outer surface and the electric field lines outside the sphere are not affected.

Electric Potential Due to a Charged Conductor

We found already that when a solid **conductor in equilibrium** carries a net charge, the charge resides on the outer surface of the conductor. Furthermore, we showed that the electric field just outside the conductor is perpendicular to the surface and that the field inside is zero.

We now show that every point on the surface of a charged conductor in equilibrium is at the same electric potential.

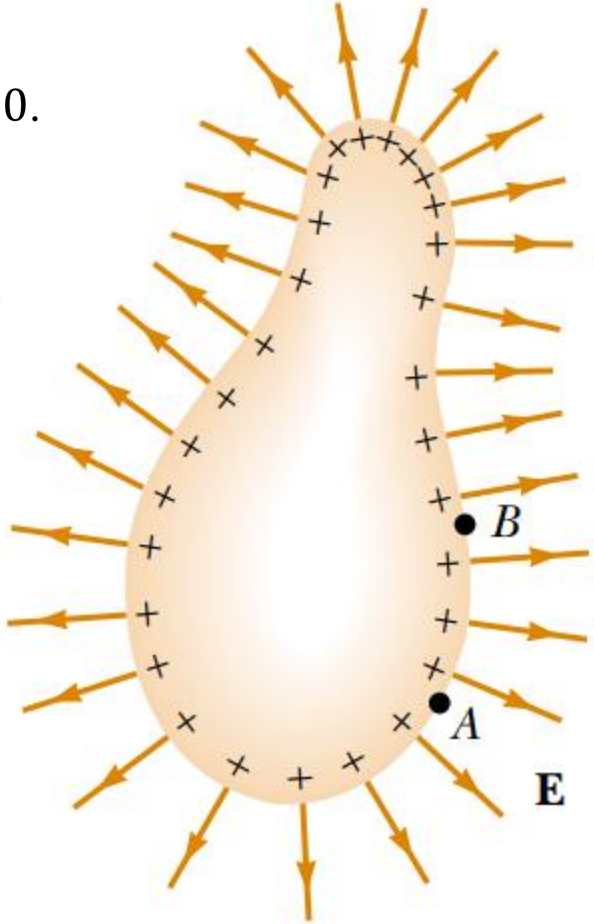
Consider two points A and B on the surface of a charged conductor. Along a surface path connecting these points, **E** is always perpendicular to the displacement d**s**; therefore $\boldsymbol{E} \cdot d\boldsymbol{s} = 0$. We conclude that the potential difference between A and B is necessarily zero:

$$V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{s} = 0$$

This result applies to any two points on the surface. Therefore, ***V*** **is constant everywhere on the surface of a charged conductor in equilibrium**. That is:

the surface of any charged conductor in electrostatic equilibrium is an equipotential surface. Furthermore, **because the electric field is zero inside the conductor, we conclude that the electric potential is constant everywhere inside the conductor and equal to its value at the surface.**

no work is required to move a test charge from the interior of a charged conductor to its surface



An arbitrarily shaped conductor carrying a positive charge. When the conductor is in electrostatic equilibrium, all of the charge resides at the surface, E = 0 inside the conductor, and the direction of E just outside the conductor is perpendicular to the surface. The electric potential is constant inside the conductor and is equal to the potential at the surface. Note from the spacing of the positive signs that the surface charge density is nonuniform.

Distribution of the free charges on a charged conductor: Peak effect

Inside a charged conductor, the Coulomb force **F**=q**E,** rearranges the electrons such that **inside** the volume of the conductor :

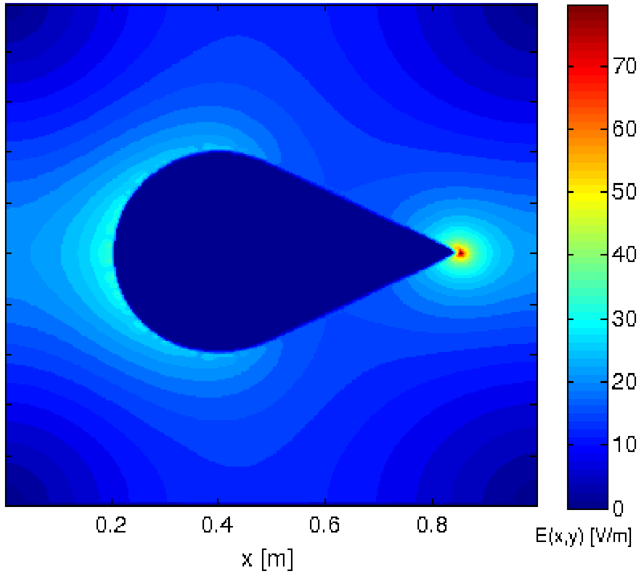
$$\mathbf{E}\left(\mathbf{r}\right)=0 \;\text{ et }\; \rho(\mathbf{r})=0$$

and on the **surface** of the conductor:

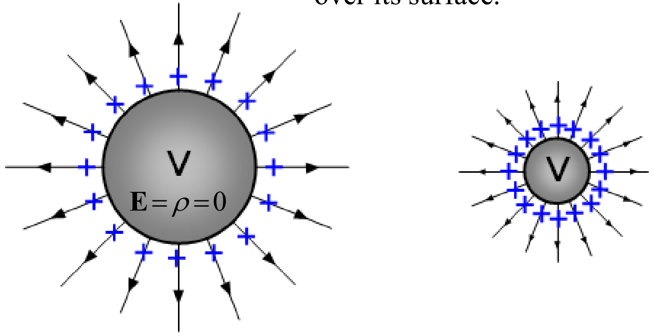
$$\mathbf{E}\left(\mathbf{r}\right)\neq 0;\;\; \sigma(\mathbf{r})\neq 0$$

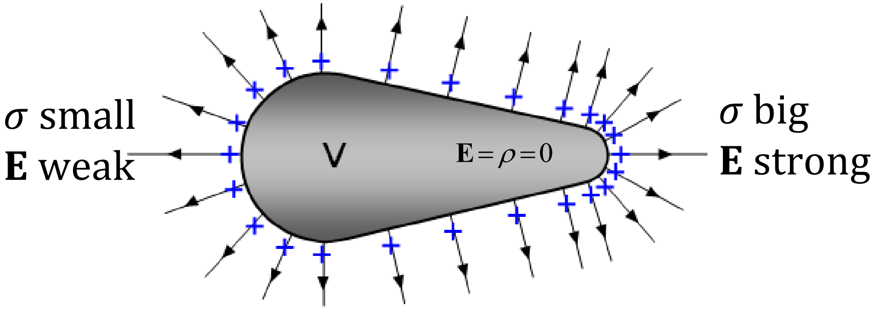
Charges try to spread out as much as possible on the surface of a conductor, and the tip of a sharp point is as far away as it is possible to be from most of the surface.
A small amount of charge on the tip can still provide a large surface density, which means a high electric field outside.

Note: It is somehow amazing that the charge of a conductor is entirely on its surface. Due to their mutual repulsion, the charges naturally spread out as much as possible, but for all of them to go to the surface seems like a waste of interior space. Well, that's just not the case. It is preferable to put all the load on the surface, and this regardless of the size or shape of the conductor. The problem can also be expressed in terms of energy. The charge on a conductor will seek the configuration that minimizes its potential energy. The electrostatic energy of a solid object is minimal when this charge is distributed over its surface.



Outside the conductor, the electric field is higher near the surface where the charge density is higher.

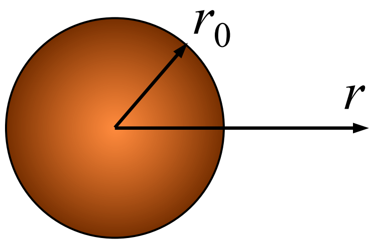


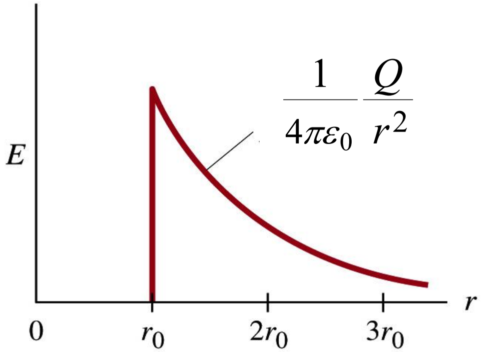


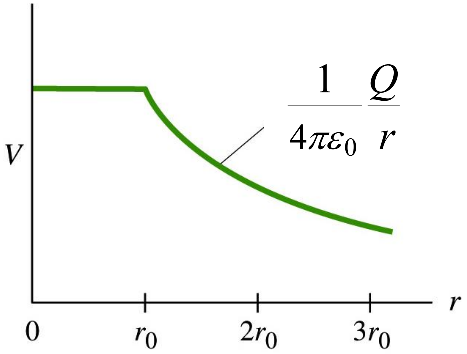
Peak effect (stronger E field near the peak):

«demonstration » by a simple case of two charged conductive spheres of different diameter

Charge "Q" on conductive sphere:

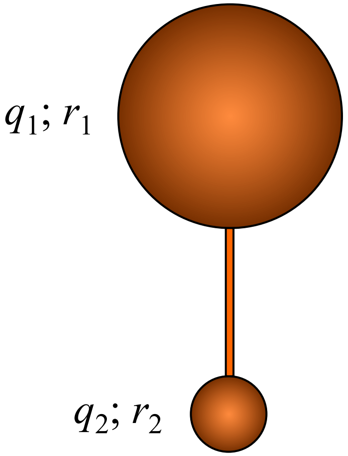






What are the electric fields E1 and E2 near the sphere's surfaces?

Charge $Q = q_1 + q_2$on two conductive spheres of different diameter:



$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} \qquad V_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2}$$

but the spheres are connected together so $V_1 = V_2 = V$ **the potentials are the same!**

$$E_1(r_1) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1^2} = \frac{V}{r_1}\ ; \quad E_2(r_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2^2} = \frac{V}{r_2}$$

$\Rightarrow$

$$\frac{E_2(r_2)}{E_1(r_1)} = \frac{r_1}{r_2} \qquad \Rightarrow E_2(r_2) > E_1(r_1)$$

mais:

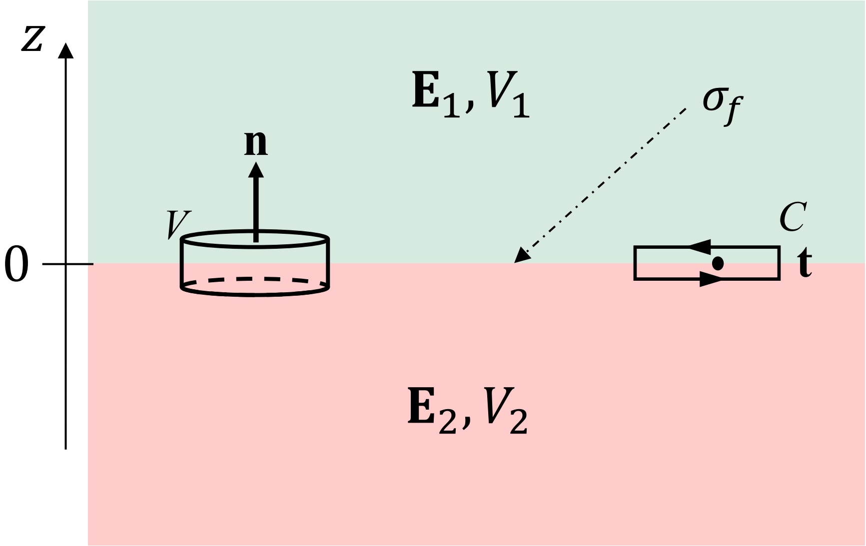
$$E_1(r_1) = \frac{\sigma_1}{\varepsilon_0}\ ; \quad E_2(r_2) = \frac{\sigma_2}{\varepsilon_0} \qquad \Rightarrow \sigma_2 > \sigma_1$$

DEMO

https://auditoires-physique.epfl.ch/experiment/593/allumage-electrostatique

The sharper is a surface (tip of a needle) the stronger is the nearby electric field

Conditions at the interface between two materials



n: normal to the separation surface between the two materials
t: tangent to the separation surface between the two materials

σ_f: Density of "free" charges on the surface ($\mathrm{C/m^2}$)

for $\sigma_f \neq 0$

From Maxwell's equations in integral form:

The electric field is discontinuous across any boundary presenting a surface charge density $\boldsymbol{\sigma_f}$

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \mathbf{n} = \sigma_f \qquad \boldsymbol{E_{2n} - E_{1n} = \sigma_f}$$

$$\Rightarrow$$

$$(\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n} = 0 \qquad E_{2t} = E_{1t}$$

$$E_{2n} = \lim_{z\to 0^-} E_{2n}(z); \quad E_{1n} = \lim_{z\to 0^+} E_{1n}(z)$$

The electric potential is, instead, always continuous across any boundary

$$\boldsymbol{V_2 = V_1}$$

$$V_2 = \lim_{z\to 0^-} V(z) = \lim_{z\to 0^+} V(z) = V_1$$

Please remember that there boundary conditions relate the fields and electric potentials just above and below the surface. So they have to be taken as limits for approaching the surface from above and from below.